The Stochastic Axion Scenario

Adam Scherlis
with Peter Graham
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See talk by Takahashi
Introduction: Axion Dark Matter

Two scenarios for the axion:

- Post-Inflationary: PQ breaks after inflation
  Precise mass needed to get axion DM
- Pre-Inflationary: PQ breaks before/during inflation
  Range of compatible masses

Usual lore: overclosure bound (or tuning/anthropics) at low mass

- Or new cosmology/axion models [Agrawal, Marques-Tavares, Xue; Nomura, Rajendran, Sanches; Dine, Fischler; Steinhardt, Turner; Lazarides, Schaefer, Seckel, Shafi; Kawasaki, Moroi, Yanagida; Dvali; Choi, Kim, Kim; Banks, Dine; Banks, Dine, Graesser]

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All masses are natural w/o new physics
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We show that there is no bound:

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<thead>
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<th>$m_a$ (eV)</th>
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Introduction: Misalignment Mechanism

After PQ breaking: axion “frozen” at $f_a \theta_0$
- Final abundance: depends on $f_a$, $\theta_0$
- Fix DM abundance: relation $f_a \leftrightarrow \theta_0$

Where does $\theta_0$ come from?
- Post-inflation PQ breaking: Temperature reaches $f_a$
  - Averaged $\theta_0$, string decay $\implies \theta_C$
  - Single $f_a$ for axion DM (“Classical Window”)
- Pre-inflation PQ breaking: Temperature stays below $f_a$
  - $\theta_0 = \mathcal{O}(1)$ implies $f_a = \mathcal{O}(10^{12} \text{ GeV})$ (“Natural”)
  - Higher $f_a$ requires smaller $\theta_0$
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Where does $\theta_0$ really come from?

Scalar field dynamics during inflation:

Every e-fold, two things happen:

- **Sliding**: classical slow-roll towards minimum
  \[
  \phi \mapsto \phi - \frac{m^2}{3H^2} \phi
  \]

- **Hopping**: quantum fluctuations ("random walk")
  \[
  \phi \mapsto \phi \pm \frac{H}{2\pi}
  \]

Eventually reaches equilibrium (independent of initial conditions and $N$)

\[
m^2 \langle \phi^2 \rangle \sim H^4
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Mechanics: Hopping and Sliding

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See PPT version for animation!
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Mechanics: Equilibrium Distribution

Fokker-Planck equation: 
\[
\dot{\rho}(\phi, t) = \frac{1}{3H_i} \partial_\phi (V'(\phi) \rho(\phi, t)) + \frac{H_i^3}{8\pi^2} \partial^2_{\phi\phi} \rho(\phi, t)
\]

- \(m^2 \langle \phi^2 \rangle \approx H_i^4\)
- Naturally small but nonzero
- \(\theta\) very uniform for individual patch

- \(H_i \leq 800\) MeV: axion has a mass
  - \(H_i \leq 200\) MeV: distribution is Gaussian
  - \(H_i > 200\) MeV: distribution is flat

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\rho(\phi) \propto \exp\left(-\frac{8\pi^2}{3H_i^4} V(\phi)\right)
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Distribution of \(\theta\) over many patches:

\[
\rho(\theta, H_i) \propto \exp \left( -\frac{8\pi^2}{3H_i^4} V(\phi) \right)
\]
Isocurvature

- Random $\mathcal{O}(H_i)$ hops build up over $> 10^{20}$ e-folds
- Inhomogeneities stretch and leave horizon
- Hops from last $\sim 60$ e-folds remain inhomogeneous

- Significant isocurvature if $H_i > 10^6$ GeV or $\theta \approx \pi \leftrightarrow f_a < 10^{10}$ GeV
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Results: Summary

For $N$ large enough to reach equilibrium:

Inflationary Axion Parameter Space

$E_i$ (GeV)

$H_i$ (GeV)

$\sim$ 10^{-5} \ 10^{-4} \ 10^{-3} \ 10^{-2} \ 10^{-1} \ 10^0 \ 10^1 \ 10^2 \ 10^3 \ 10^4 \ 10^5 \ 10^6 \ 10^7 \ 10^8 \ 10^9 \ 10^{10} \ 10^{11} \ 10^{12} \ 10^{13} \ 10^{14}$

$10^7 \ 10^8 \ 10^9 \ 10^{10} \ 10^{11} \ 10^{12} \ 10^{13} \ 10^{14} \ 10^{15} \ 10^{16}$

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kHz MHz GHz THz

$10^8$ GeV $10^9$ GeV $10^{10}$ GeV $10^{11}$ GeV $10^{12}$ GeV $10^{13}$ GeV $10^{14}$ GeV $10^{15}$ GeV $10^{16}$ GeV

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$p = .001$

$q = 10^{-100}$

$q = 10^{-1000}$

$q = 10^{-10000}$

$q = .1$

$q = .01$

$q = .001$

$q = .0001$
Results: Reheating

All of this is irrelevant if temperature hits $f_a$ during

1. Inflation: $T_{dS} \sim H_I$

2. Reheating: $T_{rh} \sim \epsilon_{\text{eff}} \sqrt{m_P H_I}$

Inefficient reheating: $T_{dS} > T_{rh}$ at high $H_I$

Note: $T_{rh} \gg \text{TeV}$ unless extremely inefficient.
Main caveat: need LOTS of inflation

- Low mass: $H_I \lesssim \Lambda_{QCD} \iff E_I \lesssim 10^9$ GeV
- Value of $H_I$ determines width of $\theta_0$ distribution
- Relaxation time: $t_{rel} = 3 \frac{H_I}{m_a^2}$ or $N_{rel} = 3 \frac{H_I^2}{m_a^2}$

Some points that naturally have the right abundance:

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See also: Guth-Takahashi-Yin [1805.08763]
(includes low-$H_I$ hilltop potential)
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See also: Guth-Takahashi-Yin [1805.08763] (includes low-$H_I$ hilltop potential)
Conclusion

Conventional claim:

But actually, unless we make assumptions about inflation:

So it’s important to search the entire mass range experimentally.
Backup: Fokker-Planck Formalism

Fokker-Planck equation: \( \dot{\rho}(\phi, t) = \frac{1}{3H_i} \partial_\phi (V'(\phi)\rho(\phi, t)) + \frac{H_i^3}{8\pi^2} \partial_{\phi\phi}^2 \rho(\phi, t) \)

Change variables,

\[ \rho(\phi, t) = \psi_0(\phi) \psi(\phi, t) \]

\[ \psi_0(\phi) := \exp(-\nu(\phi)) = \exp \left( -\frac{4\pi^2}{3H_i^4} V(\phi) \right) \]

to get Schroedinger-like equation:

\[-\frac{4\pi^2}{H_i^3} \dot{\psi}(\phi, t) = -\frac{1}{2} \psi''(\phi, t) + \frac{1}{2} \left[ -\nu''(\phi) + \nu'(\phi)^2 - \frac{3}{M_P^2} \nu(\phi) \right] \psi(\phi, t) \]

Eigenfunctions \( \rho_i = \psi_0 \psi_i \) are quasinormal modes
\( \rho_0 = \psi_0^2 \) is equilibrium distribution
Smallest positive eigenvalue is relaxation rate
Expansion rate is related to energy,

\[ 3H^2 m_P^2 = V \]

Axion contributes a small amount,

\[ V = V_I + V_a \]

\[ V_I \gg V_a \]

Regions with large \( \theta \) expand (slightly) faster

This effect suddenly becomes dominant for \( f_a \gtrsim m_P \) at \( H_I \lesssim \Lambda_{QCD} \)

Nearly all patches overproduce with \( \theta \rightarrow \pi \) (for some choice of measure)
Inflaton also has sliding and hopping
If potential is too flat, hopping dominates
\[ \implies \text{inflation becomes chaotic (eternal)} \]
Equivalent to minimum “speed” of inflaton, or maximum length of inflation

\[ N \lesssim \frac{m^2_P}{H_I^2} \]

Relaxation time violates this bound for \( f_a \gtrsim m_P \) at \( H_I \gtrsim \Lambda_{QCD} \)
Our analysis still works but eternal inflation introduces measure issues